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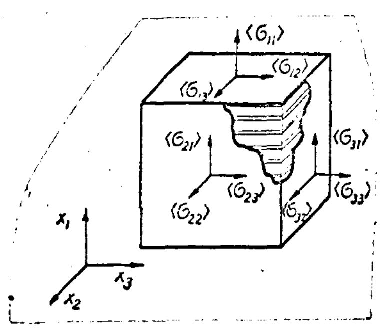
ABSTRACT

Derivation of stress-strain relations are presented for a laminar anisotropic medium from the known mechanical properties of the layers. Expressions are obtained for determining the stresses and strains in the individual layers from the known average values of the stresses and strains. The cases are examined in which the layers are elastic isotropic, elastic anisotropic or viscoelastic anisotropic. The results are applied to the calculation of the mechanical characteristics of several fiber-reinforced plastics.

1. The physical constants of materials (elastic moduli, viscous constants and others), obtained experimentally as samples with a macroscopic structure, are the mean characteristics of the physical constants of structural elements forming a particular body, which in the general case may be nonhomogeneous and anisotropic. Some of the simplest forms of structurally nonhomogeneous materials are plastics consisting of a large number of alternating homogeneous layers (see illustration). /14*



If the loads acting on such a body are such that the stresses and strains in nearly identical layers may be considered identical, such a structurally

*Numbers given in margin indicate pagination in original foreign text.

nonhomogeneous medium may be considered an averaged homogeneous anisotropic medium. The principal parameters will be the mean stresses and strains. The mean physical constants of such materials can be determined analytically by knowing the physical constants and the structure of the component elements.

We will consider an elastic body consisting of cohesive alternating isotropic layers. The Lamé constants of this body are functions of the single variable x_1 and the relationships between the stresses and strains have the form

$$\sigma_{jk} = 2\mu(x_1)\varepsilon_{jk} + \lambda(x_1)\varepsilon_{rr}\delta_{jk} \quad (j, k = 1, 2, 3). \tag{1.1}$$

Assume the body is in a homogeneous, stressed and strained state characterized by the mean tensor of stresses $\langle\sigma_{jk}\rangle$ (illustration) and the mean tensor of strains $\langle\epsilon_{jk}\rangle$. The stresses σ_{jk} and the strains ϵ_{jk} arising in the layers in this case will be functions of the one variable x_1, and therefore the equilibrium equations can be integrated; as a result we obtain

$$\sigma_{j1} = A_j \quad (j = 1, 2, 3). \tag{1.2}$$

Here A_j are the integration constants.

The stresses, strains and movements arising in the layers, to which for brevity we will apply the term "local," may be represented in the form /15

$$\begin{gathered}\sigma_{jk} = \langle\sigma_{jk}\rangle + \sigma^0_{jk}; \qquad \varepsilon_{jk} = \langle\varepsilon_{jk}\rangle + \varepsilon^0_{jk};\\ u_j = \langle\varepsilon_{jk}\rangle x_k + u^0_j; \qquad \varepsilon^0_{jk} = \frac{1}{2}(u^0_{j,1}\delta_{1k} + u^0_{k,1}\delta_{1j}) \quad (j, k = 1, 2, 3).\end{gathered} \tag{1.3}$$

where σ^0_{jk}, ϵ^0_{jk}, u^0_j are the fluctuations of the stresses, strains and movements, respectively, along the x_1 coordinate.

After substituting expressions (1.1) and (1.3) into the integrals of the equilibrium equations (1.2) we obtain the equations

$$\mu u^0_{j,1} + (\lambda + \mu) u^0_{1,1}\delta_{1j} = A_j - 2\mu\langle\varepsilon_{j1}\rangle - \lambda\langle\varepsilon_{rr}\rangle\delta_{1j} \quad (j = 1, 2, 3). \tag{1.4}$$

whose solutions have the form

$$u^0_{j,1} = \frac{A_j - 2\mu\langle\varepsilon_{j1}\rangle - \lambda\langle\varepsilon_{rr}\rangle\delta_{1j}}{(\lambda+\mu)\delta_{1j}+\mu} \quad (j = 1, 2, 3). \tag{1.5}$$

The integration constants A_j are determined from the condition of a zero value of the mean values of the fluctuations $u^0_{j,1}$

$$A_j = 2\frac{\langle\frac{\mu}{(\lambda+\mu)\delta_{1j}+\mu}\rangle}{\langle\frac{1}{(\lambda+\mu)\delta_{1j}+\mu}\rangle}\langle\varepsilon_{j1}\rangle + \frac{\langle\frac{\lambda}{(\lambda+\mu)\delta_{1j}+\mu}\rangle}{\langle\frac{1}{(\lambda+\mu)\delta_{1j}+\mu}\rangle}\langle\varepsilon_{rr}\rangle\delta_{1j}. \tag{1.6}$$

Substituting (1.5) and (1.6) into expressions (1.3), we find the values of the local stresses, strains and movements from the known mean strains. Due to the unwieldiness of these expressions we cite only those for local stresses

$$\begin{gathered}
\sigma_{11} = 2\frac{\langle\frac{\mu}{\lambda+2\mu}\rangle}{\langle\frac{1}{\lambda+2\mu}\rangle}\langle\varepsilon_{11}\rangle + \frac{\langle\frac{\lambda}{\lambda+2\mu}\rangle}{\langle\frac{1}{\lambda+2\mu}\rangle}\langle\varepsilon_{rr}\rangle;\\
\sigma_{22} = 2\mu\langle\varepsilon_{22}\rangle + 2\lambda\frac{\langle\frac{\mu}{\lambda+2\mu}\rangle - \mu\langle\frac{1}{\lambda+2\mu}\rangle}{(\lambda+2\mu)\langle\frac{1}{\lambda+2\mu}\rangle}\langle\varepsilon_{11}\rangle +\\
+\lambda\frac{\langle\frac{\lambda}{\lambda+2\mu}\rangle + 2\mu\langle\frac{1}{\lambda+2\mu}\rangle}{(\lambda+2\mu)\langle\frac{1}{\lambda+2\mu}\rangle}\langle\varepsilon_{rr}\rangle;\\
\sigma_{33} = 2\mu\langle\varepsilon_{33}\rangle + 2\lambda\frac{\langle\frac{\mu}{\lambda+2\mu}\rangle - \mu\langle\frac{1}{\lambda+2\mu}\rangle}{(\lambda+2\mu)\langle\frac{1}{\lambda+2\mu}\rangle}\langle\varepsilon_{11}\rangle +\\
+\lambda\frac{\langle\frac{\lambda}{\lambda+2\mu}\rangle + 2\mu\langle\frac{1}{\lambda+2\mu}\rangle}{(\lambda+2\mu)\langle\frac{1}{\lambda+2\mu}\rangle}\langle\varepsilon_{rr}\rangle;\\
\sigma_{12} = \frac{2}{\langle\frac{1}{\mu}\rangle}\langle\varepsilon_{12}\rangle; \qquad \sigma_{13} = \frac{2}{\langle\frac{1}{\mu}\rangle}\langle\varepsilon_{13}\rangle; \qquad \sigma_{23} = 2\mu\langle\varepsilon_{23}\rangle.
\end{gathered} \tag{1.7}$$

from which it can be seen that the movements along the x_1 coordinate will /16 be the stresses σ_{22}, σ_{33} and σ_{23}.

Averaging relations (1.7), we obtain the elastic relationships between the mean stresses and strains

$$\langle\sigma_{11}\rangle = 2\frac{\langle\frac{\mu}{\lambda+2\mu}\rangle}{\langle\frac{1}{\lambda+2\mu}\rangle}\langle\varepsilon_{11}\rangle + \frac{\langle\frac{\lambda}{\lambda+2\mu}\rangle}{\langle\frac{1}{\lambda+2\mu}\rangle}\langle\varepsilon_{rr}\rangle;$$

$$\langle\sigma_{22}\rangle = 2\langle\mu\rangle\langle\varepsilon_{22}\rangle + 2\langle\lambda\frac{\langle\frac{\mu}{\lambda+2\mu}\rangle - \mu\langle\frac{1}{\lambda+2\mu}\rangle}{(\lambda+2\mu)\langle\frac{1}{\lambda+2\mu}\rangle}\rangle\langle\varepsilon_{11}\rangle +$$

$$+\langle\lambda\frac{\langle\frac{\lambda}{\lambda+2\mu}\rangle + 2\mu\langle\frac{1}{\lambda+2\mu}\rangle}{(\lambda+2\mu)\langle\frac{1}{\lambda+2\mu}\rangle}\rangle\langle\varepsilon_{rr}\rangle; \tag{1.8}$$

$$\langle\sigma_{33}\rangle = 2\langle\mu\rangle\langle\varepsilon_{33}\rangle + 2\langle\lambda\frac{\langle\frac{\mu}{\lambda+2\mu}\rangle - \mu\langle\frac{1}{\lambda+2\mu}\rangle}{(\lambda+2\mu)\langle\frac{1}{\lambda+2\mu}\rangle}\rangle\langle\varepsilon_{11}\rangle +$$

$$+\langle\lambda\frac{\langle\frac{\lambda}{\lambda+2\mu}\rangle + 2\mu\langle\frac{1}{\lambda+2\mu}\rangle}{(\lambda+2\mu)\langle\frac{1}{\lambda+2\mu}\rangle}\rangle\langle\varepsilon_{rr}\rangle;$$

$$\langle\sigma_{12}\rangle = \frac{2}{\langle\frac{1}{\mu}\rangle}\langle\varepsilon_{12}\rangle; \qquad \langle\sigma_{13}\rangle = \frac{2}{\langle\frac{1}{\mu}\rangle}\langle\varepsilon_{13}\rangle; \qquad \langle\sigma_{23}\rangle = 2\langle\mu\rangle\langle\varepsilon_{23}\rangle.$$

On the basis of relationships (1.7) and (1.8) it is easy to find expressions for local stresses on the basis of known mean stresses.

Thus, as a result of averaging, the considered layered body is reduced to a transversally isotropic elastic medium, for which the relationships between the stresses and strains have the form (1.8). If the body consists of alternating layers, some of which have thickness h_1 and elastic constants λ_1 and μ_1, while others have the thickness h_2 and the elastic constants λ_2 and μ_2, the expressions for the mean moduli in relationships (1.8) will be as /17 follows

$$\frac{1}{\langle\frac{1}{\mu}\rangle}=\frac{(h_1+h_2)\mu_1\mu_2}{h_1\mu_2+h_2\mu_1};\qquad \frac{\langle\frac{\mu}{\lambda+2\mu}\rangle}{\langle\frac{1}{\lambda+2\mu}\rangle}=\frac{h_1\mu_1(\lambda_2+2\mu_2)+h_2\mu_2(\lambda_1+2\mu_1)}{h_1(\lambda_2+2\mu_2)+h_2(\lambda_1+2\mu_1)};$$

$$\frac{\langle\frac{\lambda}{\lambda+2\mu}\rangle}{\langle\frac{1}{\lambda+2\mu}\rangle}=\frac{h_1\lambda_1(\lambda_2+2\mu_2)+h_2\lambda_2(\lambda_1+2\mu_1)}{h_1(\lambda_2+2\mu_2)+h_2(\lambda_1+2\mu_1)};\qquad \langle\mu\rangle=\frac{h_1\mu_1+h_2\mu_2}{h_1+h_2};\tag{1.9}$$

$$\langle\lambda\frac{\langle\frac{\mu}{\lambda+2\mu}\rangle-\mu\langle\frac{1}{\lambda+2\mu}\rangle}{(\lambda+2\mu)\langle\frac{1}{\lambda+2\mu}\rangle}\rangle=-\frac{h_1h_2(\lambda_1-\lambda_2)(\mu_1-\mu_2)}{(h_1+h_2)[h_1(\lambda_2+2\mu_2)+h_2(\lambda_1+2\mu_1)]};$$

$$\langle\lambda\frac{\langle\frac{\lambda}{\lambda+2\mu}\rangle+2\mu\langle\frac{1}{\lambda+2\mu}\rangle}{(\lambda+2\mu)\langle\frac{1}{\lambda+2\mu}\rangle}\rangle=\frac{(h_1+h_2)^2\lambda_1\lambda_2+2(h_1\lambda_1+h_2\lambda_2)(h_1\mu_2+h_2\mu_1)}{(h_1+h_2)[h_1(\lambda_2+2\mu_2)+h_2(\lambda_1+2\mu_1)]}.$$

2. The layered materials actually encountered in most cases consist of anisotropic layers. These include DSP, plastics based on different fabrics, etc. In this case the elastic relationships between stresses and strains have the form

$$\sigma_{jk}=E_{jk\alpha\beta}(x_1)\varepsilon_{\alpha\beta}=E_{jk\alpha\beta}(x_1)u_{\alpha,\beta}.\tag{2.1}$$

The tensor of the elastic modulus $E_{jk\alpha\beta}$ satisfies the conditions of symmetry

$$E_{jk\alpha\beta}=E_{kj\alpha\beta};\qquad E_{jk\alpha\beta}=E_{jk\beta\alpha};\qquad E_{jk\alpha\beta}=E_{\alpha\beta jk}.\tag{2.2}$$

Substituting the relationships (2.1) into the integrals of the equilibrium equations (1.2) and taking into account the representation of (1.3), we obtain the algebraic equations for the derivatives of the fluctuations of movements

$$E_{j1\alpha1}u^0_{\alpha,1}=A_j-E_{j1\alpha\beta}\langle\varepsilon_{\alpha\beta}\rangle.\tag{2.3}$$

Their solutions will be

$$u^0_{k,1}=E^{-1}_{j1k1}(A_j-E_{j1\alpha\beta}\langle\varepsilon_{\alpha\beta}\rangle),\tag{2.4}$$

where $E^{-1}_{jk\alpha\beta}$ is a matrix, the inverse of $E_{jk\alpha\beta}$.

Since the mean values $u^0_{k,1}$ are equal to zero, we find the integration constants

$$A_j = \langle E^{-1}_{j1k1}\rangle^{-1} \langle E^{-1}_{s1k1} E_{s1\alpha\beta}\rangle \langle \varepsilon_{\alpha\beta}\rangle. \tag{2.5}$$

Then expressions (2.4) assume the form

$$u^0_{r,1} = E^{-1}_{j1r1} (\langle E^{-1}_{j1k1}\rangle \langle E^{-1}_{s1k1} E_{s1\alpha\beta}\rangle - E_{j1\alpha\beta}) \langle \varepsilon_{\alpha\beta}\rangle. \tag{2.6}$$

We find the values of the local strains and stresses from the known mean strains by substituting (2.6) into expressions (1.3) /18

$$\varepsilon_{\alpha\beta} = \left[\delta_{\alpha\sigma}\delta_{\beta\varrho} + \frac{1}{2} (E^{-1}_{j1\alpha1}\delta_{1\beta} + E^{-1}_{j1\beta1}\delta_{1\alpha}) (\langle E^{-1}_{j1k1}\rangle \langle E^{-1}_{s1k1} E_{s1\sigma\varrho}\rangle - E_{j1\sigma\varrho}) \right] \langle \varepsilon_{\sigma\varrho}\rangle; \tag{2.7}$$

$$\sigma_{rq} = \left[E_{rq\alpha\beta}\delta_{\alpha\sigma}\delta_{\beta\varrho} + \frac{1}{2} E_{rq\alpha\beta} (E^{-1}_{j1\alpha1}\delta_{1\beta} + E^{-1}_{j1\beta1}\delta_{1\alpha}) (\langle E^{-1}_{j1k1}\rangle \langle E^{-1}_{s1k1} E_{s1\sigma\varrho}\rangle - E_{j1\sigma\varrho}) \right] \langle \varepsilon_{\sigma\varrho}\rangle. \tag{2.8}$$

The relationships between the mean stresses and strains are obtained by averaging relations (2.8)

$$\langle \sigma_{rq}\rangle = \left[\langle E_{rq\alpha\beta}\rangle \delta_{\alpha\sigma}\delta_{\beta\varrho} + \frac{1}{2} \langle E_{rq\alpha\beta} (E^{-1}_{j1\sigma1}\delta_{1\beta} + E^{-1}_{j1\beta1}\delta_{1\alpha}) (\langle E^{-1}_{j1k1}\rangle \langle E^{-1}_{s1k1} E_{s1\sigma\varrho}\rangle - E_{j1\sigma\varrho})\rangle \right] \langle \varepsilon_{\sigma\varrho}\rangle. \tag{2.9}$$

3. In the same manner in which we obtained the mean values of the elastic constants of layered materials, we can obtain the mean values of the integral provisional operators describing viscoelastic properties of layered media. In this case the integrals of the equilibrium equations will be some functions of time

$$\sigma_{j1} = f_j(t) \qquad (j = 1, 2, 3). \tag{3.1}$$

The relationships between the local stresses and strains for anisotropic material are represented in the form

$$\sigma_{jk} = L_{jk\alpha\beta}(x_1)\,\varepsilon_{\alpha\beta} = L_{jk\alpha\beta}(x_1)\,u_{\alpha,\beta}, \tag{3.2}$$

where the operator $L_{jk\alpha\beta}(x_1)$ has the form

$$L_{jk\alpha\beta}(x_1) = E_{jk\alpha\beta}(x_1) + \int_0^t \varphi_{jk\alpha\beta}(x_1, t-\tau)\,d\tau \tag{3.3}$$

and has the properties of symmetry similar to (2.2).

From equations (3.1), (3.2) and the representation (1.3) we obtain

$$u^0_{k,1} = L^{-1}_{j1k1}\,[f_j(t) - L_{j1\alpha\beta}\langle\varepsilon^*_{\alpha\beta}\rangle]. \tag{3.4}$$

Here $L^{-1}_{jk\alpha\beta}$ is an operator, the inverse of $L_{jk\alpha\beta}$.

We find function $f_j(t)$ by averaging (3.4) along the x_1 coordinate

$$f_j(t) = \langle L^{-1}_{j1k1}\rangle^{-1}\langle L^{-1}_{s1k1}L_{s1\alpha\beta}\rangle\langle\varepsilon_{\alpha\beta}\rangle, \tag{3.5}$$

as a result of which expressions (3.4) assume the form

$$u^0_{r,1} = L^{-1}_{j1r1}\,(\langle L^{-1}_{j1k1}\rangle\langle L^{-1}_{s1k1}L_{s1\alpha\beta}\rangle - L_{j1\alpha\beta})\langle\varepsilon_{\alpha\beta}\rangle. \tag{3.6}$$

The expressions for local strains and stresses through the mean strains have a form similar to (2.7) and (2.8)

$$\varepsilon_{\alpha\beta} = \left[\delta_{\alpha\sigma}\delta_{\beta\varrho} + \frac{1}{2}(L^{-1}_{j1\alpha1}\delta_{1\beta} + L^{-1}_{j1\beta1}\delta_{1\alpha})(\langle L^{-1}_{j1k1}\rangle\langle L^{-1}_{s1k1}L_{s1\sigma\varrho}\rangle - L_{j1\sigma\varrho})\right]\langle\varepsilon_{\alpha\beta}\rangle; \tag{3.7}$$

$$\begin{aligned}\sigma_{rq} = \Big[L_{rq\alpha\beta}\delta_{\alpha\sigma}\delta_{\beta\varrho} &+ \frac{1}{2}L_{rq\alpha\beta}(L^{-1}_{j1\alpha1}\delta_{1\beta} + \\ &+ L^{-1}_{j1\beta1}\delta_{1\alpha})(\langle L^{-1}_{j1k1}\rangle\langle L^{-1}_{s1k1}L_{s1\sigma\varrho}\rangle - L_{j1\sigma\varrho})\Big]\langle\varepsilon_{\sigma\varrho}\rangle.\end{aligned} \tag{3.8}$$

Averaging relationships (3.8) along the x_1 coordinate, we obtain the /19

operator relationships of viscoelasticity between the mean stresses and the mean strains

$$\langle\sigma_{rq}\rangle = \Big[\langle L_{rq\alpha\beta}\rangle\,\delta_{\alpha\sigma}\delta_{\beta\varrho} + \frac{1}{2}\langle L_{rq\alpha\beta}\langle L^{-1}_{j1\alpha1}\delta_{1\beta} + L^{-1}_{j1\beta1}\delta_{1\alpha}\rangle(\langle L^{-1}_{j1k1}\rangle\langle L^{-1}_{s1k1}L_{s1\sigma\varrho}\rangle - L_{j1\sigma\varrho})\rangle\Big]\langle\varepsilon_{\sigma\varrho}\rangle. \quad (3.9)$$

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